Identification of space deformation using linear and superficial quadratic variations

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Abstract

We use linear and superficial quadratic variations to identify a bijective space deformation that makes a non-stationary Gaussian random field stationary.

keywords: identification of a model; mean square convergence; quadratic variations; space deformation; stationary reducibility

1 Introduction

In spatial statistics we are often concerned with non-stationary phenomena. For most applications dealing with a non-stationary Gaussian random field, the first step in classical approaches consists of removing expectation, dividing the residual by the standard deviation and modelling the residual as a stationary process. That is to say, the random field $Y = \{Y(x, y) : (x, y) \in G \subseteq \mathbb{R}^2\}$ (where G stands for Geographical space) under study is of the form

$$Y(x, y) = \mu(x, y) + \sigma(x, y)Z(x, y),$$

where $\mu(x, y) = EY(x, y)$, $\sigma(x, y)^2 = E(Y(x, y) - \mu(x, y))^2$ and $Z(x, y)$ is a reduced (centred and standardised) stationary Gaussian random field. Then the non-stationarity of the random field Y is simply understood as non-stationarity of both the first order moment $\mu(x, y)$ and the standard deviation $\sigma(x, y)$. Nevertheless, Z can also be non-stationary. In this case, Sampson and Guttorp [11] propose a transformation of the index space G with a bijective space deformation. Formally, it consists of modelling $Z(x, y)$ as

$$Z(x, y) = \delta(\Phi(x, y)), \tag{1}$$

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where δ is stationary and $\Phi = (\Phi_1, \Phi_2) : G \longrightarrow D$ (where D stands for Deformed space) is a bijective deformation. Equivalently, the correlation function r of Z is

$$r(x, y, x', y') = R\left(\Phi(x', y') - \Phi(x, y)\right), \tag{2}$$

where R is a stationary correlation function in $\mathbb{R}^2$.

In this paper, we consider a reduced Gaussian random field Z indexed by $G = [0,1]^2$ satisfying (1)-(2) and we suppose that the stationary correlation R is known. Our concern is the functional estimation of the space deformation Φ using a set of suitable linear and superficial quadratic variations of Z.

The quadratic variations are first introduced by Lévy [8] who shows that if Z is the standard Wiener process on $[0,1]$, then almost surely its quadratic variation on $[0,1]$ converges to 1. Baxter [2] and further Gladyshev [3] generalise this result to a large class of Gaussian processes. Guyon and León [5] introduce the H-variations for stationary Gaussian processes, a generalisation of these quadratic variations. They study the convergence in distribution of the H-variations, suitably normalised.

For Gaussian process Z with stationary increments, Istas and Lang [6] define general quadratic variations, substituting a general discrete difference operator to the simple difference $Z(k/n) - Z((k-1)/n)$. They use these quadratic variations to estimate the Hölder index of a process.

For non-stationary Gaussian processes, with increments stationary or not, Perrin [9] gives a general result concerning the functional asymptotic normality of the process of the quadratic variations which corresponds to the linear interpolation of the points $(p/n, V_n(p/n))$, $p = 1, 2, \ldots, n$, with $V_n(p/n)$ the discrete quadratic variations at points p/n. This result is applied to the estimation of a time deformation for non-stationary models of the form $Z(x) = \delta(\Phi(x)),\quad x \in [0,1]$.

The generalisation of quadratic variations for stationary Gaussian fields is studied in Guyon [4] and León and Ortega [7]. Another generalisation for non-stationary Gaussian processes and quadratic variations along curves is done in Adler and Pyke [1]. Guyon [4] shows that some stationary random fields can be identified in mean square sense using different families of variations. Using some of these families allows us, in this paper, to generalise the result of [9] to the estimation of a space deformation for non-stationary models of the form (1)-(2).

The paper is structured as follows. Section 2 sets up notations, assumptions, definitions and describes the superficial and the linear quadratic variations. In Section 3, we study the pointwise mean square (L_2) convergence of these quadratic variations. Finally in Section 4, we propose an estimator of Φ which converges in L_2 to Φ. This estimator is defined with the help of the superficial and the linear quadratic variations.

2 Linear and Superficial quadratic variations

Let $Z = \{Z(x, y), (x, y) \in [0,1]^2\}$ be a real-valued reduced Gaussian random field with correlation r satisfying (2).

For any differentiable function $f : (x,y) \in [0,1]^2 \to \mathbb{R}$, we denote using $f^{(p_1,p_2)}$ the p_1,p_2-partial derivative of f with respect to x and y. Assume for the stationary correlation function R

(A1) $R(u,v)$ satisfies when $u \to 0$ and $v \to 0$:
$R(u,v) = 1 - \alpha|u| - \beta|v| + O(uv), \quad$ where $\alpha > 0$ and $\beta > 0$.

(A2) $R^{(2,0)}(u,v), R^{(1,1)}(u,v), R^{(0,2)}(u,v)$ exist and are uniformly bounded in $\{(u,v) : u \neq 0\}, \{(u,v) : uv \neq 0\}, \{(u,v) : v \neq 0\}$.

For instance, the stationary exponential model $R(u,v) = \exp(-\alpha|u| - \beta|v|)$ satisfies **(A1)**-**(A2)**.

We consider the following smoothness assumption for the deformation Φ

(B1) Φ has uniformly bounded second derivatives in $[0,1]^2$.

We restrict to bijective deformations $\Phi = (\Phi_1, \Phi_2)$ such that Φ_1 and Φ_2 satisfy the following assumptions for all $(x,y) \in [0,1]^2$

(B2) $\Phi_1^{(1,0)}(x,y) > 0, \Phi_2^{(0,1)}(x,y) > 0, \Phi_1^{(0,1)}(x,y) \geq 0, \Phi_2^{(1,0)}(x,y) \geq 0.$

Moreover, it is reasonable to assume that the Jacobian determinant of Φ is strictly positive in $[0,1]^2$. In particular, it follows from **(B2)** that for all $(x,y) \in [0,1]^2$ we have

$$\frac{\Phi_1^{(0,1)}(x,y)}{\Phi_1^{(1,0)}(x,y)} < \frac{\Phi_2^{(0,1)}(x,y)}{\Phi_2^{(1,0)}(x,y)}.$$

We strengthen this condition by

(B3) $a = \sup_{(x,y)\in[0,1]^2} \frac{\Phi_1^{(0,1)}(x,y)}{\Phi_1^{(1,0)}(x,y)} < \inf_{(x,y)\in[0,1]^2} \frac{\Phi_2^{(0,1)}(x,y)}{\Phi_2^{(1,0)}(x,y)} = b.$

It is easy to see (*cf.* [10]) that if (Φ, R) is a solution to (2), then any other solution $(\tilde{\Phi}, \tilde{R})$ is of the form $\tilde{\Phi}(x) = B\Phi(x,y) + b$ and $\tilde{R}(u) = R(B^{-1}u)$, where B is a regular square matrix and b is a vector in $\mathbb{R}^2$. Thus, without loss of generality, we may impose that

$$\Phi(0) = 0. \tag{3}$$

Bijections $\Phi = (\Phi_1, \Phi_2)$ such that $\Phi_1(x,y) = F(x)$ and $\Phi_2(x,y) = G(y)$ for all $(x,y) \in [0,1]^2$, where F and G are two twice continuously differentiable strictly increasing functions in $[0,1]$, as are F^{-1} and G^{-1}, satisfy **(B1)**-**(B3)** with $a = 0$ and $b = \infty$. More generally, twice continuously differentiable bijections $\Phi = (\Phi_1, \Phi_2)$ in $[0,1]^2$, as are Φ^{-1}, with strictly positive first partial derivatives and where Φ_1 does not depend on y (respectively Φ_2 does not depend on x) satisfy **(B1)**-**(B3)** with $a = 0$ (respectively $b = \infty$). Bijections of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} c_1x + c_2y + c_3xy \\ d_1x + d_2y + d_3xy \end{pmatrix},$$

where $c_1 > 0$, $c_2 \geq 0$, $c_3 \geq 0$, $d_1 \geq 0$, $d_2 > 0$, $d_3 \geq 0$ and $(c_2+c_3)(d_1+d_3) < c_1 d_2$, satisfy **(B1)**-**(B3)**. These deformations transform lines onto lines and reduce to linear transformations when $c_3 = d_3 = 0$. In this case Z is already a stationary process with correlation function $\tilde{R}(u,v) = R((u,v)M)$ where $M = \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix}$.

Let n and m be two positive integers, $\Pi_{n,m}$ the product partition of $[0,1]^2$ with mesh $1/n$ in x and mesh $1/m$ in y. We call $\lambda = \dfrac{m}{n}$ the geometry of $\Pi_{n,m}$. Note that λ is a parameter under our control. We define the rectangular increment $Z(\Delta) = Z(x',y') - Z(x',y) - Z(x,y') + Z(x,y)$ for the rectangle $\Delta = [(x,y),(x',y')[$, with $0 \leq x < x' \leq 1$ and $0 \leq y < y' \leq 1$. Then we define for $k = 0,1,\ldots,n-1$ and $y \in [0,1]$

$$\Delta_{k,y} = [(k/n, \lfloor m(y \wedge \frac{m-1}{m}) \rfloor / m), ((k+1)/n, (\lfloor m(y \wedge \frac{m-1}{m}) \rfloor + 1)/m)[,$$

where $\lfloor u \rfloor$ denotes the greatest integer smaller than or equal to u, and $u \wedge v$ denotes the minimum of u and v. We define $\Delta_{x,l}$ in a symmetric way for $l = 0,1,\ldots,m-1$ and $x \in [0,1]$

$$\Delta_{x,l} = [(\lfloor n(x \wedge \frac{n-1}{n}) \rfloor / n, l/m), ((\lfloor n(x \wedge \frac{n-1}{n}) \rfloor + 1)/n, (l+1)/m)[.$$

We also consider the following one-dimensional Horizontal and Vertical increments for $k = 0,1,\ldots,n-1$ and $l = 0,1,\ldots,m-1$

$$Z(\Delta_k^H) = Z\left(\frac{k+1}{n}, 0\right) - Z\left(\frac{k}{n}, 0\right) \text{ and } Z(\Delta_l^V) = Z\left(0, \frac{l+1}{m}\right) - Z\left(0, \frac{l}{m}\right).$$

We define for all $(x,y) \in [0,1]^2$ two superficial quadratic variations, $H_{n,\lambda}(x,y)$ (in x at fixed level y) and $V_{m,\lambda}(x,y)$ (in y at fixed level x) as follows

$$H_{n,\lambda}(x,y) = \sum_{k=0}^{\lfloor nx \rfloor - 1} (Z(\Delta_{k,y}))^2 \text{ and } V_{m,\lambda}(x,y) = \sum_{l=0}^{\lfloor my \rfloor - 1} (Z(\Delta_{x,l}))^2.$$

We also define two linear quadratic variations $h_n(x)$, $x \in [0,1]$, and $v_m(y)$, $y \in [0,1]$, as follows

$$h_n(x) = \sum_{k=0}^{\lfloor nx \rfloor - 1} \left(Z(\Delta_k^H)\right)^2 \text{ and } v_m(y) = \sum_{l=0}^{\lfloor my \rfloor - 1} \left(Z(\Delta_l^V)\right)^2.$$

When $\lfloor nx \rfloor = 0$ (respectively $\lfloor my \rfloor = 0$) we set $H_{n,\lambda}(x,y) = 0$ and $h_n(x) = 0$ (respectively $V_{m,\lambda}(x,y) = 0$ and $v_m(y) = 0$).

3 Asymptotic properties

In this section, we are concerned with asymptotic properties of the quadratic variations defined in the previous section. Because of the symmetry in the definitions of $H_{n,\lambda}(x,y)$ and $V_{m,\lambda}(x,y)$ (respectively $h_n(x)$ and $v_m(y)$), we focus our attention on $H_{n,\lambda}(x,y)$ and $h_n(x)$.

3.1 Convergence of the means

3.1.1 Superficial quadratic variations

Denote using $\mathbb{Q}^+$ the set of the strictly positive rational numbers. Define for all $(x,y) \in [0,1]^2$ and any $\lambda \in \mathbb{Q}^+$

$$\begin{aligned} H_\lambda(x,y) &= 4\left(\beta(\Phi_2(x,y) - \Phi_2(0,y)) + \frac{\alpha}{\lambda}\int_0^x \Phi_1^{(0,1)}(u,y)du\right), \\ V_\lambda(x,y) &= 4\left(\beta\lambda\int_0^y \Phi_2^{(1,0)}(x,v)dv + \alpha(\Phi_1(x,y) - \Phi_1(x,0))\right). \end{aligned}$$

We first establish the following lemma

Lemma 3.1 *Assume **(B1)-(B3)**. Then for all $(x,y) \in [0,1]^2$ such that $(x+1/n, y+1/m) \in [0,1]^2$ and any $\lambda \in]a,b[\cap\mathbb{Q}^+$*

$$\Phi_1(x+1/n,y) \geq \Phi_1(x,y+1/m) \text{ and } \Phi_2(x,y+1/m) \geq \Phi_2(x+1/n,y),$$

as $n \to \infty$ and $m \to \infty$.

Proof. Under assumption **(B1)**, a Taylor expansion of order one with Lagrange remainder gives for all $(x,y) \in [0,1]^2$ such that $(x+1/n, y+1/m) \in [0,1]^2$

$$\Phi_1(x+1/n,y) - \Phi_1(x,y+1/m) = \frac{1}{n}\Phi_1^{(1,0)}(x,y) - \frac{1}{m}\Phi_1^{(0,1)}(x,y) + O\left(\frac{1}{m^2}\right),$$

where $O(.)$ is uniform in both x and y. Then from **(B2)** and **(B3)** we obtain

$$\Phi_1(x+1/n,y) - \Phi_1(x,y+1/m) \geq \frac{1}{m}\Phi_1^{(1,0)}(x,y)(\lambda - a) + O\left(\frac{1}{m^2}\right).$$

Thus, for any $\lambda \in]a,b[\cap\mathbb{Q}^+$, $\Phi_1(x+1/n,y) - \Phi_1(x,y+1/m) \geq 0$ as $m \to \infty$.

Under assumptions **(B1)**-**(B3)**, we obtain for all $(x,y) \in [0,1]^2$ such that $(x+1/n, y+1/m) \in [0,1]^2$

$$\Phi_2(x,y+1/m) - \Phi_2(x+1/n,y) \geq \frac{1}{n}\Phi_2^{(0,1)}(x,y)\left(\frac{1}{\lambda} - \frac{1}{b}\right) + O\left(\frac{1}{n^2}\right),$$

where $O(.)$ is uniform in both x and y. Thus, for any $\lambda \in]a,b[\cap\mathbb{Q}^+$, $\Phi_2(x,y+1/m) - \Phi_2(x+1/n,y) \geq 0$ as $n \to \infty$.

□

Theorem 3.1 *Assume **(A1)** and **(B1)-(B3)**. Then for all $(x,y) \in [0,1]^2$ and any $\lambda \in]a,b[\cap\mathbb{Q}^+$*

$$(i) \lim_{n\to\infty} E\left(H_{n,\lambda}(x,y)\right) = H_\lambda(x,y) \text{ and } (ii) \lim_{m\to\infty} E\left(V_{m,\lambda}(x,y)\right) = V_\lambda(x,y).$$

Proof.

(i) We set $y' = y \wedge \dfrac{m-1}{m}$ for all $y \in [0,1]$ and

$$\begin{array}{ll} A = \big(k/n, \lfloor my' \rfloor/m\big), & B = \big((k+1)/n, \lfloor my' \rfloor/m\big), \\ C = \big(k/n, (\lfloor my' \rfloor + 1)/m\big), & D = \big((k+1)/n, (\lfloor my' \rfloor + 1)/m\big), \end{array}$$

for $k = 0, 1, \dots, n-1$. We have for all $(x,y) \in [0,1]^2$: $E\left(H_{n,\lambda}(x,y)\right) = \sum_{k=0}^{\lfloor nx \rfloor - 1} E\left(Z(\Delta_{k,y})\right)^2$ with

$$\begin{aligned} E\left(Z(\Delta_{k,y})\right)^2 = 4 + 2\{&R\left(\Phi(D) - \Phi(A)\right) + R\left(\Phi(B) - \Phi(C)\right) - R\left(\Phi(B) - \Phi(A)\right) \\ &- R\left(\Phi(C) - \Phi(A)\right) - R\left(\Phi(D) - \Phi(B)\right) - R\left(\Phi(D) - \Phi(C)\right)\}. \end{aligned}$$

From **(A1)**, **(B2)** and Lemma (3.1) we get as $n \to \infty$ and $m \to \infty$

$$\begin{aligned} E\left(Z(\Delta_{k,y})\right)^2 &= 2\alpha\left(\Phi_1(D) - \Phi_1(A) - \Phi_1(B) + \Phi_1(C)\right) \\ &+ 2\beta\left(\Phi_2(D) - \Phi_2(A) + \Phi_2(B) - \Phi_2(C)\right) \\ &+ O\left((\Phi_1(D) - \Phi_1(A))(\Phi_2(D) - \Phi_2(A))\right). \end{aligned}$$

Under assumption **(B1)**, a Taylor expansion of order one for both Φ_1 and Φ_2 gives

$$\begin{aligned} &E\left(Z(\Delta_{k,y})\right)^2 \\ &\quad = \frac{4}{n}\left(\beta\Phi_2^{(1,0)}\left(\frac{k}{n}, \frac{\lfloor my' \rfloor}{m}\right) + \frac{\alpha}{\lambda}\Phi_1^{(0,1)}\left(\frac{k}{n}, \frac{\lfloor my' \rfloor}{m}\right)\right) + O\left(\frac{1}{nm}\right), \end{aligned} \tag{4}$$

where $O(.)$ is uniform in both k and y. In addition, a Taylor expansion of order 0 gives

$$\begin{aligned} \Phi_1^{(0,1)}\left(\frac{k}{n}, \frac{\lfloor my' \rfloor}{m}\right) &= \Phi_1^{(0,1)}\left(\frac{k}{n}, y\right) + O\left(\frac{1}{m}\right), \\ \Phi_2^{(1,0)}\left(\frac{k}{n}, \frac{\lfloor my' \rfloor}{m}\right) &= \Phi_2^{(1,0)}\left(\frac{k}{n}, y\right) + O\left(\frac{1}{m}\right). \end{aligned}$$

Thus

$$E\left(H_{n,\lambda}(x,y)\right) = \frac{4}{n}\sum_{k=0}^{\lfloor nx \rfloor - 1}\left(\beta\Phi_2^{(1,0)}\left(\frac{k}{n}, y\right) + \frac{\alpha}{\lambda}\Phi_1^{(0,1)}\left(\frac{k}{n}, y\right)\right) + O\left(\frac{1}{n}\right). \tag{5}$$

Since $\Phi_2^{(1,0)}(., y)$ and $\Phi_1^{(0,1)}(., y)$ are Riemann integrable in $[0,1]$ for all $y \in [0,1]$, we get for all $(x,y) \in [0,1]^2$ and for any $\lambda \in]a, b[\cap \mathbb{Q}^+$

$$\lim_{n \to \infty} E\left(H_{n,\lambda}(x,y)\right) = 4\left(\beta(\Phi_2(x,y) - \Phi_2(0,y)) + \frac{\alpha}{\lambda}\int_0^x \Phi_1^{(0,1)}(u,y)du\right). \tag{6}$$

(ii) We get for all $(x,y) \in [0,1]^2$

$$E\left(V_{m,\lambda}(x,y)\right) = \frac{4}{m}\sum_{l=0}^{\lfloor my \rfloor - 1}\left(\lambda\beta\Phi_2^{(1,0)}\left(x, \frac{l}{m}\right) + \alpha\Phi_1^{(0,1)}\left(x, \frac{l}{m}\right)\right) + O\left(\frac{1}{m}\right),$$

so that for any $\lambda \in]a,b[\cap \mathbb{Q}^+$

$$\lim_{m\to\infty} E\left(V_{m,\lambda}(x,y)\right) = 4\left(\beta\lambda\int_0^y \Phi_2^{(1,0)}(x,v)dv + \alpha(\Phi_1(x,y) - \Phi_1(x,0))\right).$$

$\square$

3.1.2 Linear quadratic variations

Define for all $(x,y) \in [0,1]^2$

$$h(x) = 2\left(\alpha\Phi_1(x,0) + \beta\Phi_2(x,0)\right) \text{ and } v(y) = 2\left(\alpha\Phi_1(0,y) + \beta\Phi_2(0,y)\right).$$

Theorem 3.2 *Assume **(A1)** and **(B1)-(B2)**. Then for all $(x,y) \in [0,1]^2$*

$$(i)\ \lim_{n\to\infty} E\left(h_n(x)\right) = h(x) \text{ and } (ii)\ \lim_{m\to\infty} E\left(v_m(y)\right) = v(y).$$

Proof.

(i) For all $x \in [0,1]$: $E\left(h_n(x)\right) = \sum_{k=0}^{\lfloor nx\rfloor - 1} E\left(Z(\Delta_k^H)\right)^2$. From **(A1)** and **(B1)-(B2)** we have as $n \to \infty$

$$E\left(Z(\Delta_k^H)\right)^2 = 2\alpha\Phi_1\left(\Delta_k^H\right) + 2\beta\Phi_2\left(\Delta_k^H\right) + O\left(\Phi_1\left(\Delta_k^H\right)\Phi_2\left(\Delta_k^H\right)\right).$$

A Taylor expansion of order one for both Φ_1 and Φ_2 gives

$$E\left(Z(\Delta_k^H)\right)^2 = \frac{2}{n}\left(\alpha\Phi_1^{(1,0)}\left(\frac{k}{n},0\right) + \beta\Phi_2^{(1,0)}\left(\frac{k}{n},0\right)\right) + O\left(\frac{1}{n^2}\right), \tag{7}$$

where $O(.)$ is uniform in k. Therefore, we get for all $x \in [0,1]$

$$E\left(h_n(x)\right) = \frac{2}{n}\sum_{k=0}^{\lfloor nx\rfloor - 1}\left(\alpha\Phi_1^{(1,0)}\left(\frac{k}{n},0\right) + \beta\Phi_2^{(1,0)}\left(\frac{k}{n},0\right)\right) + O\left(\frac{1}{n}\right). \tag{8}$$

$\Phi_1^{(1,0)}(.,0)$ and $\Phi_2^{(1,0)}(.,0)$ being Riemann integrable in $[0,1]$, it follows from (3) that the right-hand side on (8) converges to $2\left(\alpha\Phi_1(x,0) + \beta\Phi_2(x,0)\right)$ as $n \to \infty$.

A similar treatment holds for *(ii)*.

$\square$

3.2 L_2 convergence of the variations

3.2.1 Superficial quadratic variations

Theorem 3.3 *Assume **(A1)-(A2)** and **(B1)-(B3)**. Then for all $(x,y) \in [0,1]^2$ and any $\lambda \in]a,b[\cap \mathbb{Q}^+$*

$$(i)\ \lim_{n\to\infty} H_{n,\lambda}(x,y) \overset{L_2}{=} H_\lambda(x,y) \text{ and } (ii)\ \lim_{m\to\infty} V_{m,\lambda}(x,y) \overset{L_2}{=} V_\lambda(x,y).$$

Proof. We have to prove that the variance $var\,(H_{n,\lambda}(x,y))$ converges to 0 as $n \to \infty$. For $(k,k') \in \{0,1,\dots,n-1\}^2$ and $y \in [0,1]$ set

$$c_{k,k'}(y) = E\left(Z(\Delta_{k,y})Z(\Delta_{k',y})\right). \tag{9}$$

Then $var\,(H_{n,\lambda}(x,y)) = 2\sum_{k=0}^{\lfloor nx\rfloor -1}\sum_{k'=0}^{\lfloor nx\rfloor -1} c_{k,k'}(y)$, this equality coming from, for $(\xi_1,\xi_2,\xi_3,\xi_4)$ a centred Gaussian vector

$$E(\xi_1\xi_2\xi_3\xi_4) = E(\xi_1\xi_2)E(\xi_3\xi_4) + E(\xi_1\xi_3)E(\xi_2\xi_4) + E(\xi_1\xi_4)E(\xi_2\xi_3). \tag{10}$$

Therefore

$$var\,(H_{n,\lambda}(x,y)) = 2\sum_{k=0}^{\lfloor nx\rfloor -1} c_{k,k}^2(y) + 4\sum_{k=0}^{\lfloor nx\rfloor -1}\sum_{k'>k} c_{k,k'}^2(y). \tag{11}$$

Since the derivatives of Φ are uniformly bounded in $[0,1]^2$, we get from (4) $c_{k,k}(y) = O\left(n^{-1}\right)$, where $O(.)$ is uniform in both k and y. It follows that the first term on the right-hand side of (11) converges to 0 as $n \to \infty$. It remains to prove that the second term converges to 0 as well. We have for any y in $[0,1]$

$$c_{k,k'}(y) = \sum_{i,j,i'j' \in \{0,1\}} (-1)^{i+j+i'+j'} r\left(\frac{k+i}{n}, \frac{\lfloor my\rfloor + j}{m}, \frac{k'+i'}{n}, \frac{\lfloor my\rfloor + j'}{m}\right).$$

For any differentiable function $g : (x,y,x',y') \in [0,1]^4 \to \mathbb{R}$, we denote using $g^{(p_1,p_2,p_3,p_4)}$ the p_1,p_2,p_3,p_4-partial derivative of g with respect to x, y, x' and y'. Let A be a bound for all the quantities $|r^{(p_1,p_2,p_3,p_4)}(x,y,x',y')|$, $p_1+p_2+p_3+p_4 = 2$, in the range $(x,y) \neq (x',y')$. Under assumptions **(A2)** and **(B1)**, using for $r(x,y,x',y')$ a Taylor series expansion of order 1, it can easily be shown that $k' \neq k$ implies

$$|c_{k,k'}(y)| \leq \frac{4A}{n^2}\left(3 + \frac{3}{\lambda^2} + \frac{4}{\lambda}\right).$$

It follows that the second term on the right-hand side of (11) converges to 0 as $n \to \infty$. Therefore, the left-hand side of (11) converges to 0.

□

3.2.2 Linear quadratic variations

Theorem 3.4 *Assume* ***(A1)-(A2)*** *and* ***(B1)-(B2)****. Then for all* $(x,y) \in [0,1]^2$

$$(i)\ \lim_{n\to\infty} h_n(x) \stackrel{L_2}{=} h(x) \ \text{and}\ (ii)\ \lim_{m\to\infty} v_m(y) \stackrel{L_2}{=} v(y).$$

Proof. We have to prove that $var\,(h_n(x))$ converges to 0 as $n \to \infty$. For $(k,k') \in \{0,1,\dots,n-1\}^2$ set

$$d_{k,k'} = E\left(Z(\Delta_k^H)Z(\Delta_{k'}^H)\right), \tag{12}$$

then due to (10)

$$var\left(h_n(x)\right) = 2\sum_{k=0}^{\lfloor nx\rfloor-1}\sum_{k'=0}^{\lfloor nx\rfloor-1} d_{k,k'} = 2\sum_{k=0}^{\lfloor nx\rfloor-1} d_{k,k}^2 + 4\sum_{k=0}^{\lfloor nx\rfloor-1}\sum_{k'>k} d_{k,k'}^2. \qquad (13)$$

Since the derivatives of Φ are uniformly bounded in $[0,1]^2$, we get from (7) $d_{k,k} = O\left(n^{-1}\right)$, where $O(.)$ is uniform in k. It follows that the first term on the right-hand side of (13) converges to 0 as $n \to \infty$. It remains to prove that the second term converges to 0 as well. We have

$$d_{k,k'} = \sum_{i,j\in\{0,1\}} (-1)^{i+j} r\left(\frac{k+i}{n}, 0, \frac{k'+j}{n}, 0\right).$$

Under assumptions **(A2)** and **(B1)**, a Taylor series expansion or order 1 gives for $k' \neq k$

$$|d_{k,k'}| \leq \frac{3A}{n^2}.$$

It follows that the second term on the right-hand side of (13) converges to 0 as $n \to \infty$. Therefore, the left-hand side of (13) converges to 0.

□

4 Estimator of the space deformation

Using the fact that λ, the geometry of the rectangular partition $\Pi_{n,m}$, is a parameter under our control, the superficial quadratic variations, $H_{n,\lambda}(x,y)$ and $V_{m,\lambda}(x,y)$, for two distinct λ's, together with the linear quadratic variations, $h_n(x)$ and $v_m(y)$, provide a useful tool for identifying the space deformation Φ in model (1)-(2). Let us define for all $(x,y) \in [0,1]^2$ and any two distinct values λ_1 and λ_2 of λ in $]a,b[\cap\mathbb{Q}^+$

$$\begin{aligned}
\hat{\Phi}_{1,n}(x,y) &= \frac{\lambda_1 V_{\lambda_2 n,\lambda_2}(x,y) - \lambda_2 V_{\lambda_1 n,\lambda_1}(x,y) + 2(\lambda_1-\lambda_2)h_n(x)}{4\alpha(\lambda_1-\lambda_2)} \\
&- \frac{(\lambda_1 H_{n,\lambda_1}(x,0) - \lambda_2 H_{n,\lambda_2}(x,0))}{4\alpha(\lambda_1-\lambda_2)}, \\
\hat{\Phi}_{2,n}(x,y) &= \frac{\lambda_1 H_{n,\lambda_1}(x,y) - \lambda_2 H_{n,\lambda_2}(x,y) + 2(\lambda_1-\lambda_2)v_n(y)}{4\beta(\lambda_1-\lambda_2)} \\
&- \frac{(\lambda_1 V_{\lambda_2 n,\lambda_2}(0,y) - \lambda_2 V_{\lambda_1 n,\lambda_1}(0,y))}{4\beta(\lambda_1-\lambda_2)}.
\end{aligned}$$

Here is our main theorem

Theorem 4.1 *Assume **(A1)-(A2)** and **(B1)-(B3)**. Then $\hat{\Phi}_n = (\hat{\Phi}_{1,n}, \hat{\Phi}_{2,n})$ converges in L_2 to Φ as $n \to \infty$.*

***Proof*.** In the sequel, all the convergences are in L_2. It follows from Theorem 3.3-*(i)* that for any $(\lambda_1, \lambda_2) \in (]a, b[\cap \mathbb{Q}^+)^2$

$$\lim_{n\to\infty} (\lambda_1 H_{n,\lambda_1}(x,y) - \lambda_2 H_{n,\lambda_2}(x,y)) = 4\beta(\lambda_1 - \lambda_2)(\Phi_2(x,y) - \Phi_2(0,y)). \quad (14)$$

Due to (3) we deduce by setting $y = 0$

$$\lim_{n\to\infty} (\lambda_1 H_{n,\lambda_1}(x,0) - \lambda_2 H_{n,\lambda_2}(x,0)) = 4\beta(\lambda_1 - \lambda_2)\Phi_2(x,0).$$

Using Theorem 3.4-*(i)* we then have

$$\lim_{n\to\infty} (2(\lambda_1 - \lambda_2)h_n(x) - (\lambda_1 H_{n,\lambda_1}(x,0) - \lambda_2 H_{n,\lambda_2}(x,0))) = 4\alpha(\lambda_1 - \lambda_2)\Phi_1(x,0). \quad (15)$$

Similarly, we have from Theorem 3.3-*(ii)*

$$\lim_{m\to\infty} (V_{m,\lambda_1}(x,y)/\lambda_1 - V_{m,\lambda_2}(x,y)/\lambda_2) = 4\alpha(1/\lambda_1 - 1/\lambda_2)(\Phi_1(x,y) - \Phi_1(x,0)). \quad (16)$$

Due to (3) we deduce by setting $x = 0$

$$\lim_{m\to\infty} (V_{m,\lambda_1}(0,y)/\lambda_1 - V_{m,\lambda_2}(0,y)/\lambda_2) = 4\alpha(1/\lambda_1 - 1/\lambda_2)\Phi_1(0,y).$$

Using Theorem 3.4-*(ii)* we then have

$$\lim_{m\to\infty} (2(1/\lambda_1 - 1/\lambda_2)v_m(y) - (V_{m,\lambda_1}(0,y)/\lambda_1 - V_{m,\lambda_2}(0,y)/\lambda_2)) = 4\beta(1/\lambda_1 - 1/\lambda_2)\Phi_2(0,y). \quad (17)$$

Therefore, from (15) and (16) and from (14) and (17) we obtain $\hat{\Phi}_n = (\hat{\Phi}_{1,n}, \hat{\Phi}_{2,n})$ as an estimator which converges to Φ in L_2 as $n \to \infty$.

□

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