Reducing non-stationary random fields to stationarity or isotropy using a space deformation

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Abstract

Recently, a model for non-stationary random field $Z = \{Z(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n, n \geq 2\}$ has been developed. This consists of reducing Z to stationarity or isotropy *via* a bijective bi-continuous deformation Φ of the index space. We give the form of this deformation under smoothness assumptions.

keywords: correlation function; stationary reducibility; isotropic reducibility; weak stationarity

1 Introduction

In spatial statistics we are often concerned with non-stationary phenomena. For most applications dealing with a non-stationary random field, the first step in classical approaches consists of removing expectation, dividing the residual by the standard deviation and modelling the residual as a stationary process. The random field $Z = \{Z(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^n\}$ under study is of the form

$$Z(\mathbf{x}) = \mu(\mathbf{x}) + \sigma(\mathbf{x})\epsilon(\mathbf{x})$$

where $\mu(\mathbf{x}) = EZ(\mathbf{x})$, $\sigma^2(\mathbf{x}) = E(Z(\mathbf{x}) - \mu(\mathbf{x}))^2$ and $\epsilon(\mathbf{x})$ is a centred and standardised weakly (or strongly) stationary random field. The non-stationarity of the random field Z is then understood as non-stationarity of both the first order moment $\mu(\mathbf{x})$ and the variance $\sigma^2(\mathbf{x})$.

This question is studied by Sampson and Guttorp (1992) who propose to transform the index space $\mathbb{R}^2$ with a bijective space deformation. Sampson and Guttorp's approach finds its origin in multidimensional scaling techniques. Formally, it consists of modelling $\epsilon(\mathbf{x})$ as

$$\epsilon(\mathbf{x}) = \delta(\Phi(\mathbf{x}))$$

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where δ is weakly isotropic, and Φ is a bijective deformation, or equivalently of modelling the correlation function $r(\mathbf{x}, \mathbf{y})$ of the random field Z as

$$r(\mathbf{x}, \mathbf{y}) = \rho(\|\Phi(\mathbf{y}) - \Phi(\mathbf{x})\|) \tag{1}$$

where ρ is an isotropic correlation function and $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

When a correlation function satisfies the model (1) we call it an isotropic reducible (*ir*) correlation function. Unlike the classical approaches, non-stationarity through second order moments is thus taken into account and model (1) gives an opportunity to enlarge the class of models for studying non-stationary random fields.

Meiring (1995) gives numerous developments in the modelling and estimation of non-stationary correlation structure using space deformation. Perrin and Meiring (1998) study the uniqueness of (Φ, ρ) in (1) under different conditions. With this method, many sets of environmental data have been analysed: solar radiation (Sampson and Guttorp, 1992), acid precipitation (Guttorp *et al.*, 1992, 1993; Guttorp and Sampson, 1994; Mardia and Goodall, 1993), rainfall (Monestiez *et al.*, 1993), air pollution (Brown *et al.*, 1994) and tropospheric ozone (Sampson *et al.*, 1994; Guttorp *et al.*, 1994; Meiring, 1995) *etc.* In the one-dimensional case, $n = 1$, Perrin and Senoussi (1998) give a characterisation of the non-stationary correlation functions that can be reduced to stationarity via a differentiable deformation.

Sampson and Guttorp (1992) refer only to isotropic reducibility. In this paper we study stationarity reducibility (*sr*) as well, that is, we consider correlation functions $r(\mathbf{x}, \mathbf{y})$ such that

$$r(\mathbf{x}, \mathbf{y}) = R(\Phi(\mathbf{y}) - \Phi(\mathbf{x})) \tag{2}$$

where R is a stationary correlation function. Indeed, there are random fields which are *sr* but not *ir*. For instance, this is the case for Brownian sheets. Their correlation function is of the form for all $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$

$$r(\mathbf{x}, \mathbf{y}) = \frac{\|\mathbf{x}\| + \|\mathbf{y}\| - \|\mathbf{y} - \mathbf{x}\|}{2\|\mathbf{x}\|\|\mathbf{y}\|}. \tag{3}$$

A direct calculation using the polar coordinates shows that they satisfy (2) with $\Phi = (\Phi_1, \Phi_2)$ defined by

$$\Phi_1(\mathbf{x}) = \ln(\|\mathbf{x}\|) \quad \text{and} \quad \Phi_2(\mathbf{x}) = \arctan(x_2/x_1), \quad \mathbf{x} = (x_1, x_2),$$

and R defined by

$$R(u_1, u_2) = \frac{1}{2}\left(\exp(\frac{u_1}{2}) + \exp(-\frac{u_1}{2}) - \sqrt{\exp(\frac{u_1}{2}) + \exp(-\frac{u_1}{2}) - 2\cos(u_2)}\right).$$

In this paper, we characterise *sr* and *ir* correlation functions under smoothness assumptions. The paper is organised as follows. In Section 2 we give some properties of the model (2) and propose a characterisation for smooth *sr* correlation functions in the form of a system of partial differential equations. Further, we prove uniqueness of the deformation Φ up to a bijective affine transformation. For some particular cases, we give the form of the deformation that makes a non-stationary random field stationary. Section 3 characterises smooth *ir* correlation functions for which we give a characterisation. We prove uniqueness of the deformation up to a homothetic Euclidean motion. Our main result, theorem 3.3, gives the general form of the deformation that reduces a non-stationary random field to isotropy, under smoothness assumptions. Section 4 briefly discusses the possibility of applying our work to second order moment functions other than correlation functions.

2 Stationary reducibility

2.1 Properties.

So far, we have considered correlation functions. However the model (2) can be applied to covariance functions provided that the variance of the process is constant. Moreover, when the mean of the process is also constant, the model (2) generalises the notion of weak stationarity (or strong stationarity for Gaussian random fields); indeed, when the deformation Φ is the identity function, stationarity and *sr* agree.

2.2 Characterisation.

We suppose that Φ is a bijective deformation from $\mathbb{R}^n$ onto $\mathbb{R}^n$ satisfying the following assumption

(A1) Φ is differentiable in $\mathbb{R}^n$ as is its inverse.

Let J_Φ denote the Jacobian matrix of Φ and set for any differentiable function $f : (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^n \longmapsto f(\mathbf{x}, \mathbf{y}) \in \mathbb{R}$

$$\begin{cases} D_{\mathbf{x}} f(\mathbf{x},\mathbf{y}) &= (\partial_1 f, \dots, \partial_n f)(\mathbf{x},\mathbf{y}) \\ D_{\mathbf{y}} f(\mathbf{x},\mathbf{y}) &= (\partial_{n+1} f, \dots, \partial_{n+n} f)(\mathbf{x},\mathbf{y}), \end{cases}$$

where $\partial_i f(\mathbf{x}, \mathbf{y})$, $i = 1, 2, \dots, 2n$, denotes the i^{th} first partial derivative of $f(\mathbf{x}, \mathbf{y})$. We denote by $\mathbf{0}$ the zero vector of $\mathbb{R}^n$ and by $(w_1, w_2, \dots, w_n)$ a column vector in $\mathbb{R}^n$. We consider correlation functions $r(\mathbf{x}, \mathbf{y})$ such that

(A2) $r(\mathbf{x}, \mathbf{y})$ is continuously differentiable for $\mathbf{x} \neq \mathbf{y}$.

First let us characterise the stationary correlation functions.

Lemma 2.1 *Let $C(\mathbf{x}, \mathbf{y})$ be a correlation function which is differentiable for $\mathbf{x} \neq \mathbf{y}$. $C(\mathbf{x}, \mathbf{y})$ is stationary, i.e. there is a stationary correlation function R such that*

$$C(\mathbf{x}, \mathbf{y}) = R(\mathbf{y} - \mathbf{x}), \tag{4}$$

if and only if the following holds

$$D_{\mathbf{x}}C(\mathbf{x},\mathbf{y}) + D_{\mathbf{y}}C(\mathbf{x},\mathbf{y}) = \mathbf{0}, \quad \mathbf{x} \neq \mathbf{y}. \tag{5}$$

Proof. The necessity is obvious. Conversely assume (5) holds, consider the bijective coordinate change $\mathbf{u} = \mathbf{y} - \mathbf{x}$ and $\mathbf{v} = \mathbf{y} + \mathbf{x}$ and set $\Gamma(\mathbf{u},\mathbf{v}) = C\left(\mathbf{x}(\mathbf{u},\mathbf{v}), \mathbf{y}(\mathbf{u},\mathbf{v})\right)$. Equation (5) then leads to $D_{\mathbf{v}}\Gamma(\mathbf{u},\mathbf{v}) = \mathbf{0}$. Thus Γ does not depend on the coordinates $\mathbf{v}$ so that we may define $R(\mathbf{u}) = \Gamma(\mathbf{u},\mathbf{v})$ and (4) follows.

□

This leads to a necessary and sufficient condition for stationary reducibility *via* a bijective deformation.

Theorem 2.1 *Assume **(A1)** and **(A2)** hold. Then r satisfies (2) if and only if*

$$D_{\mathbf{x}}r(\mathbf{x},\mathbf{y})J_{\Phi}^{-1}(\mathbf{x}) + D_{\mathbf{y}}r(\mathbf{x},\mathbf{y})J_{\Phi}^{-1}(\mathbf{y}) = \mathbf{0}, \quad \mathbf{x} \neq \mathbf{y}. \tag{6}$$

Proof. We set $\gamma(\mathbf{u},\mathbf{v}) = r\left(\Phi^{-1}(\mathbf{u}), \Phi^{-1}(\mathbf{v})\right)$. Thus, r satisfies (2) if and only if γ satisfies (4). It follows from lemma 2.1 that r satisfies (2) if and only if

$$D_{\mathbf{u}}\gamma(\mathbf{u},\mathbf{v}) + D_{\mathbf{v}}\gamma(\mathbf{u},\mathbf{v}) = \mathbf{0}, \quad \mathbf{u} \neq \mathbf{v},$$

with

$$\begin{cases} D_{\mathbf{u}}\gamma(\mathbf{u},\mathbf{v}) &= D_{\mathbf{x}}r\left(\Phi^{-1}(\mathbf{u}), \Phi^{-1}(\mathbf{v})\right) J_{\Phi}^{-1}(\Phi^{-1}(\mathbf{u})) \\ D_{\mathbf{v}}\gamma(\mathbf{u},\mathbf{v}) &= D_{\mathbf{y}}r\left(\Phi^{-1}(\mathbf{u}), \Phi^{-1}(\mathbf{v})\right) J_{\Phi}^{-1}(\Phi^{-1}(\mathbf{v})). \end{cases}$$

To conclude we set $\mathbf{x} = \Phi^{-1}(\mathbf{u})$ and $\mathbf{y} = \Phi^{-1}(\mathbf{v})$.

□

2.3 Uniqueness.

If (Φ, R) is a solution to (2), then for any regular square matrix A and any vector b, $(\tilde{\Phi}, \tilde{R})$ with $\tilde{\Phi}(\mathbf{x}) = A\Phi(\mathbf{x}) + b$ and $\tilde{R}(\mathbf{u}) = R(A^{-1}\mathbf{u})$ is a solution as well. Thus, without loss of generality we may impose the restriction that

$$\Phi(\mathbf{0}) = \mathbf{0} \text{ and } J_{\Phi}(\mathbf{0}) = Id \tag{7}$$

where Id denotes the identity matrix in $\mathbb{R}^n$. Then, when the non-stationary correlation function $r(\mathbf{x},\mathbf{y})$ satisfying (2) and the deformation Φ are given, the stationary correlation function R is uniquely determined as

$$R(\mathbf{u}) = r(\mathbf{0}, \Phi^{-1}(\mathbf{u})). \tag{8}$$

It follows from **(A1)** and **(A2)** that the stationary correlation function $R(\mathbf{u})$ is continuously differentiable for $\mathbf{u}$ different from $\mathbf{0}$. Uniqueness of Φ will be proved under the following regularity assumption

(A3) the set of functions $\{\mathbf{y} \neq \mathbf{0} \longmapsto \partial_j r(\mathbf{0}, \mathbf{y}), j = 1, 2, \ldots, n\}$ is linearly independent.

Theorem 2.2 *Assume (A1), (A2) and (A3) hold. If (Φ, R) is a solution to (2), with Φ satisfying conditions (7), then it is unique.*

Proof. Let (Φ_1, R_1) and (Φ_2, R_2) be two solutions to (2), Φ_1 satisfying $\Phi_1(\mathbf{0}) = \mathbf{0}$, $J_{\Phi_1}(\mathbf{0}) = Id$, and Φ_2 satisfying $\Phi_2(\mathbf{0}) = \mathbf{0}$, $J_{\Phi_2}(\mathbf{0}) = Id$. Set $\Theta = \Phi_2 \circ \Phi_1^{-1}$ and $\Gamma(\mathbf{u}, \mathbf{v}) = r(\Phi_1^{-1}(\mathbf{u}), \Phi_1^{-1}(\mathbf{v}))$, then

$$\Gamma(\mathbf{u}, \mathbf{v}) = R_1(\mathbf{v} - \mathbf{u}) = R_2(\Theta(\mathbf{v}) - \Theta(\mathbf{u})).$$

It follows from theorem 2.1 that the following relations hold for all $\mathbf{u} \neq \mathbf{v}$

$$\begin{cases} D_{\mathbf{u}}\Gamma(\mathbf{u}, \mathbf{v}) & = & -D_{\mathbf{v}}\Gamma(\mathbf{u}, \mathbf{v}) \\ D_{\mathbf{u}}\Gamma(\mathbf{u}, \mathbf{v}) & = & -D_{\mathbf{v}}\Gamma(\mathbf{u}, \mathbf{v}) J_{\Theta}^{-1}(\mathbf{v}) J_{\Theta}(\mathbf{u}), \end{cases}$$

from which we deduce

$$D_{\mathbf{u}}\Gamma(\mathbf{u}, \mathbf{v}) = D_{\mathbf{u}}\Gamma(\mathbf{u}, \mathbf{v}) J_{\Theta}^{-1}(\mathbf{v}) J_{\Theta}(\mathbf{u}), \quad \mathbf{u} \neq \mathbf{v}.$$

Set $\mathbf{x} = \Phi_1^{-1}(\mathbf{u})$ and $\mathbf{y} = \Phi_1^{-1}(\mathbf{v})$. Then for all $\mathbf{x} \neq \mathbf{y}$

$$D_{\mathbf{x}} r(\mathbf{x}, \mathbf{y}) J_{\Phi_1}^{-1}(\mathbf{x}) = D_{\mathbf{x}} r(\mathbf{x}, \mathbf{y}) J_{\Phi_1}^{-1}(\mathbf{x}) J_{\Theta}^{-1}(\Phi_1(\mathbf{y})) J_{\Theta}(\Phi_1(\mathbf{x})).$$

When $\mathbf{x} = \mathbf{0}$, $J_{\Phi_1}^{-1}(\mathbf{0}) = J_{\Theta}(\Phi_1(\mathbf{0})) = Id$ and the previous equality becomes

$$D_{\mathbf{x}} r(\mathbf{0}, \mathbf{y}) = D_{\mathbf{x}} r(\mathbf{0}, \mathbf{y}) J_{\Theta}^{-1}(\Phi_1(\mathbf{y})), \quad \mathbf{y} \neq \mathbf{0}. \tag{9}$$

Under assumption **(A3)** equation (9) implies that $J_{\Theta}(\Phi_1(\mathbf{y})) = Id$ for all $\mathbf{y} \neq \mathbf{0}$, *i.e.* $\Phi_1 = \Phi_2$. Finally, we deduce from (8) that $R_1 = R_2$.

□

Corollary 2.1 *Assume (A1), (A2) and (A3) hold and suppose (Φ, R) is a solution to (2). Then any other solution $(\tilde{\Phi}, \tilde{R})$ is of the form*

$$\tilde{\Phi}(\mathbf{x}) = A\Phi(\mathbf{x}) + b \ \ \textit{and} \ \ \tilde{R}(\mathbf{u}) = R(A^{-1}\mathbf{u}).$$

2.4 Solution for some particular types of deformations.

In general, equation (6) has no solution, that is, we can not give the form of the deformation in (2) without any additional assumption. However, when we impose the deformation Φ to belong to a particular family of bijections, it is possible to solve (6). Hereafter, we consider three different families of bijections.

In the following examples we suppose that assumptions **(A1)**, **(A2)** and **(A3)** hold.

2.4.1 A special deformation of the plane.

When $n = 2$, we consider the bijective deformations $\Phi = (\phi_1, \phi_2)$ of the plane that satisfy the Cauchy's conditions, *i.e.* $\partial_1\phi_1 = \partial_2\phi_2$ and $\partial_1\phi_2 = -\partial_2\phi_1$. Impose the restrictions $\Phi(\mathbf{0}) = \mathbf{0}$ and $J_\Phi(\mathbf{0}) = Id$. By setting $\mathbf{y} = \mathbf{0}$ in (6), we then get

$$\left\{\begin{array}{lcl} \partial_1 r(\mathbf{x},\mathbf{0}) & = & -\partial_3 r(\mathbf{x},\mathbf{0})\partial_1\phi_1 + \partial_4 r(\mathbf{x},\mathbf{0})\partial_2\phi_1 \\ \partial_2 r(\mathbf{x},\mathbf{0}) & = & -\partial_3 r(\mathbf{x},\mathbf{0})\partial_2\phi_1 - \partial_4 r(\mathbf{x},\mathbf{0})\partial_1\phi_1, \end{array}\right.$$

from which we deduce

$$\left\{\begin{array}{lcl} \partial_1\phi_1 & = & -\dfrac{\partial_1 r\partial_3 r + \partial_2 r\partial_4 r}{(\partial_3 r)^2 + (\partial_4 r)^2}(\mathbf{x},\mathbf{0}) \\ \\ \partial_2\phi_1 & = & \dfrac{\partial_1 r\partial_4 r - \partial_2 r\partial_3 r}{(\partial_3 r)^2 + (\partial_4 r)^2}(\mathbf{x},\mathbf{0}). \end{array}\right.$$

After integrating along the path $s \in [0,1] \longmapsto s\mathbf{x}$, with $\mathbf{x} = (x_1, x_2)$, we get

$$\left\{\begin{array}{lcl} \phi_1(x_1,x_2) & = & \displaystyle\int_0^1 \frac{-x_1(\partial_1 r\partial_3 r + \partial_2 r\partial_4 r) + x_2(\partial_1 r\partial_4 r - \partial_2 r\partial_3 r)}{(\partial_3 r)^2 + (\partial_4 r)^2}(s\mathbf{x},0)ds \\ \\ \phi_2(x_1,x_2) & = & \displaystyle\int_0^1 \frac{-x_2(\partial_1 r\partial_3 r + \partial_2 r\partial_4 r) + x_1(\partial_2 r\partial_3 r - \partial_1 r\partial_4 r)}{(\partial_3 r)^2 + (\partial_4 r)^2}(s\mathbf{x},0)ds. \end{array}\right.$$

2.4.2 Independent deformations in each direction.

Set $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$. When $\Phi = (\phi_1, \phi_2, \dots, \phi_n)$ with $\phi_i(\mathbf{x}) = \phi_i(x_i)$ for all $\mathbf{x} \in \mathbb{R}^n$, $i = 1, 2, \dots, n$, we infer from (6) for $i = 1, 2, \dots, n$

$$\partial_i r(\mathbf{x},\mathbf{y})(\phi_i)'(y_i) + \partial_{n+i} r(\mathbf{x},\mathbf{y})(\phi_i)'(x_i) = 0, \quad x_i \neq y_i,$$

where $(\phi_i)'$ denotes the first derivative of ϕ_i. Let $(e_1, \dots, e_n)$ be the canonical basis of $\mathbb{R}^n$. Thus, the deformation Φ satisfying (7) is defined by

$$\phi_i(x_i) = -\int_0^{x_i} \frac{\partial_i r(se_i,\mathbf{0})}{\partial_{n+i} r(se_i,\mathbf{0})}ds, \quad i = 1, 2, \dots, n.$$

2.4.3 Space deformations which leave linear manifolds invariant.

When Φ is of type $\Phi(\mathbf{x}) = \mathbf{x}\varphi(\mathbf{x})$ with φ a real-valued function, then $J_\Phi(\mathbf{x}) = \varphi(\mathbf{x})I_n + \mathbf{x}J_\varphi(\mathbf{x})$. If we impose $\varphi(\mathbf{0}) = 1$, then Φ satisfies (7) and from (6) we obtain for $i = 1, 2, \dots, n$ and $\mathbf{x} \neq \mathbf{0}$

$$\partial_i r(\mathbf{x},\mathbf{0}) + \varphi(\mathbf{x})\partial_{n+i} r(\mathbf{x},\mathbf{0}) + \partial_i\varphi(\mathbf{x})\left(\sum_{j=1}^n x_j\partial_{n+j} r(\mathbf{x},\mathbf{0})\right) = 0.$$

Multiply each previous equation by x_i, sum over i and set $p(\mathbf{x}) = -\left(\sum_{j=1}^n x_j\partial_j r\right) / \left(\sum_{j=1}^n x_j\partial_{n+j} r\right)(\mathbf{x},\mathbf{0})$. Then, we get the non-homogeneous linear partial differential equation for $\mathbf{x} \neq \mathbf{0}$

$$\varphi(\mathbf{x}) + \sum_{i=1}^{n} x_i \partial_i \varphi(\mathbf{x}) = p(\mathbf{x}). \tag{10}$$

For any direction $d_0 \in \mathbb{R}^n$ and for all $s \in \mathbb{R}$ we set $\mathbf{x} = sd_0$, $f(s) = \varphi(sd_0)$ and $q(s) = p(sd_0)$. Then (10) reduces to the non-homogeneous linear differential equation for $s \neq 0$

$$f(s) + sf'(s) = q(s).$$

Therefore, $sf(s) = \displaystyle\int_0^s q(t)dt$. Thus, with $\varphi(\mathbf{0}) = 1$ the unique solution satisfy for all s

$$\Phi(sd_0) = d_0 \int_0^s p(td_0)du.$$

Note that stationarity already exists in the direction d_0 when $p(sd_0) \equiv 1$.

3 Isotropic reducibility

3.1 Properties.

As for the *sr* case, the model (1) can be applied to covariance functions provided that the variance is constant. Moreover, when the mean is also constant, the model (1) generalises the notion of weak isotropy (or strong isotropy for Gaussian random fields); indeed, when the deformation Φ is the identity function, isotropy and *ir* agree.

3.2 Characterisation.

First let us characterise the isotropic correlation functions.

Lemma 3.1 *Let $C(\mathbf{x}, \mathbf{y})$ be a correlation function which is differentiable for $\mathbf{x} \neq \mathbf{y}$. $C(\mathbf{x}, \mathbf{y})$ is isotropic, i.e. there is an isotropic correlation function ρ such that*

$$C(\mathbf{x}, \mathbf{y}) = \rho(\|\mathbf{y} - \mathbf{x}\|) \tag{11}$$

if and only if (5) holds and

$$y_i \partial_j C(\mathbf{0}, \mathbf{y}) = y_j \partial_i C(\mathbf{0}, \mathbf{y}), \quad \mathbf{y} \neq \mathbf{0} \ \textit{ and } \ 1 \leq i < j \leq n. \tag{12}$$

Proof. First we prove that a function $R(\mathbf{x})$ on $\mathbb{R}^n$ depends only on the norm $\|\mathbf{x}\|$

$$R(\mathbf{x}) = \rho(\|\mathbf{x}\|) \tag{13}$$

if and only if

$$x_i \partial_j R(\mathbf{x}) = x_j \partial_i R(\mathbf{x}), \quad \mathbf{x} \neq \mathbf{0} \ \text{ and } \ 1 \leq i < j \leq n. \tag{14}$$

If (13) holds, $\partial_i R(\mathbf{x}) = \rho'(\|\mathbf{x}\|)x_i/\|x\|$, so that (14) is satisfied. Conversely assume (14) holds, express $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in the spherical coordinates $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$

$$\begin{cases} x_1 &= \theta_1 \cos\theta_2 \\ x_2 &= \theta_1 \sin\theta_2 \cos\theta_3 \\ \vdots &= \vdots \\ x_n &= \theta_1 \sin\theta_2 \ldots \sin\theta_n, \end{cases}$$

and set $L(\theta_1, \theta_2, \ldots, \theta_n) = R(\mathbf{x}(\theta_1, \theta_2, \ldots, \theta_n))$. Then we have

$$x_1 \partial_{\theta_i} L(\theta) = x_1 \sum_{j=i}^{n} \partial_j R(\mathbf{x}) \partial_{\theta_i} x_j(\theta), \quad i = 2, 3, \ldots, n.$$

It follows from (14) that $x_1 \partial_j R(\mathbf{x}) = x_j \partial_1 R(\mathbf{x})$ so that for $i = 2, 3, \ldots, n$

$$x_1 \partial_{\theta_i} L(\theta) = \partial_1 R(\mathbf{x}) \sum_{j=1}^{n} x_j \partial_{\theta_i} x_j(\theta) = (1/2)\partial_1 R(\mathbf{x}) \partial_{\theta_i} \sum_{j=1}^{n} (x_j)^2 = 0.$$

The last equality is due to the fact that $\sum_{j=1}^{n}(x_j)^2 = \theta_1^2$ is independent of θ_i, $i = 2, 3, \ldots, n$. Thus L does not depend of the θ_i's, $i = 2, 3, \ldots, n$. Therefore we may define $\rho(\theta_1) = L(\theta_1, \theta_2, \ldots, \theta_n)$.

Finally, $C(\mathbf{x}, \mathbf{y})$ satisfies (11) if and only if $C(\mathbf{x}, \mathbf{y}) = R(\mathbf{y} - \mathbf{x})$ with $R(\mathbf{y} - \mathbf{x})$ satisfying

$$(y_i - x_i)\partial_j R(\mathbf{y} - \mathbf{x}) = (y_j - x_j)\partial_i R(\mathbf{y} - \mathbf{x}), \quad \mathbf{x} \neq \mathbf{y} \text{ and } 1 \leq i < j \leq n. \quad (15)$$

To conclude observe that $C(\mathbf{x}, \mathbf{y}) = C(\mathbf{0}, \mathbf{y} - \mathbf{x})$ and $\partial_i R(\mathbf{y} - \mathbf{x}) = -\partial_i C(\mathbf{0}, \mathbf{y} - \mathbf{x})$, $i = 1, 2, \ldots, n$, and set $\mathbf{x} = \mathbf{0}$ in (15).

□

Note that if (Φ, ρ) is a solution to (1), then for any vector b, $(\tilde{\Phi}, \rho)$ with $\tilde{\Phi} = \Phi + b$ is a solution as well. Thus, without loss of generality, we may impose in the sequel the restriction that $\Phi(\mathbf{0}) = \mathbf{0}$.

We set $\Phi = (\phi_1, \phi_2, \ldots, \phi_n)^t$ and we denote by $c_i(\mathbf{0})$ the $i-$th column vector of $J_\Phi^{-1}(\mathbf{0})$, $i = 1, 2, \ldots, n$. Here is the necessary and sufficient condition for isotropic reducibility *via* a bijective bi-differentiable deformation.

Theorem 3.1 *Assume **(A1)** and **(A2)** hold. Then r satisfies (1) if and only if (6) holds and*

$$\phi_i(\mathbf{y}) D_{\mathbf{x}} r(\mathbf{0}, \mathbf{y}) c_j(\mathbf{0}) = \phi_j(\mathbf{y}) D_{\mathbf{x}} r(\mathbf{0}, \mathbf{y}) c_i(\mathbf{0}), \quad \mathbf{y} \neq \mathbf{0} \text{ and } 1 \leq i < j \leq n. \quad (16)$$

Proof. We set $\gamma(\mathbf{u}, \mathbf{v}) = r\left(\Phi^{-1}(\mathbf{u}), \Phi^{-1}(\mathbf{v})\right)$. Thus, r satisfies (1) if and only if γ satisfies (11). It follows from lemma 3.1 that γ satisfies (11) if and only if it satisfies (5) and

$$v_i \partial_j \gamma(\mathbf{0}, \mathbf{v}) = v_j \partial_i \gamma(\mathbf{0}, \mathbf{v}), \quad \mathbf{v} \neq \mathbf{0} \text{ and } 1 \leq i < j \leq n,$$

with $\partial_i \gamma(\mathbf{0}, \mathbf{v}) = D_{\mathbf{x}} r(\Phi^{-1}(\mathbf{0}), \Phi^{-1}(\mathbf{v})) c_i(\Phi^{-1}(\mathbf{0}))$ and $\Phi^{-1}(\mathbf{0}) = \mathbf{0}$. To conclude we set $\mathbf{y} = \Phi^{-1}(\mathbf{v})$.

$\square$

Remark 3.1 *Clearly, equations (16) are still true when $i = j$.*

As already mentioned in the introduction, the correlation function (3) of the Brownian sheets is *sr* with a stationary correlation function R defined by

$$R(u_1, u_2) \\ = \frac{1}{2}\left(\exp(u_1/2) + \exp(-u_1/2) - \sqrt{\exp(u_1/2) + \exp(-u_1/2) - 2\cos(u_2)}\right).$$

Calculation of the partial derivatives of $R(u_1, u_2)$ shows that equations (12) are not satisfied. Thus, the correlation function of the Brownian sheets is not *ir*.

3.3 Uniqueness.

Let $G(n)$ be the group of the regular square matrices of dimension n and $S(n)$ the subgroup of the orthogonal matrices. For a symmetric and positive definite matrix A which has the decomposition $A = PDP^t$, where P is orthogonal and D is diagonal, we note $A^{1/2} = PD^{1/2}P^t$.

Theorem 3.2 *Assume **(A1)**, **(A2)** and **(A3)** hold, and suppose (Φ, ρ) is a solution to (1). Then any other solution $(\tilde{\Phi}, \tilde{\rho})$ is of the form*

$$\tilde{\Phi}(\mathbf{x}) = aP\Phi(\mathbf{x}) \quad \textit{and} \quad \tilde{\rho}(t) = \rho(t/a)$$

where a is a strictly positive scalar and $P \in S(n)$.

Proof. Obviously, if (Φ, ρ) is a solution to (1) then for any orthogonal matrix P and any scalar $a > 0$, $(\tilde{\Phi}, \tilde{\rho})$ with $\tilde{\Phi}(\mathbf{x}) = aP\Phi(\mathbf{x})$ and $\tilde{\rho}(t) = \rho(t/a)$ is a solution as well.

Conversely, if Φ and $\tilde{\Phi}$ are solutions to (1), they are solutions to (2). It follows from corollary (2.1) that $\tilde{\Phi}(\mathbf{x}) = A\Phi(\mathbf{x})$, where A is a regular square matrix. The matrix A^tA is symmetric and positive definite, and thus admits the decomposition $A^tA = PDP^t$, where P is orthogonal and D is diagonal. We have $\rho(\|\Phi(\mathbf{y}) - \Phi(\mathbf{x})\|) = \tilde{\rho}(\|\tilde{\Phi}(\mathbf{y}) - \tilde{\Phi}(\mathbf{x})\|)$. We set $\mathbf{x} = \mathbf{0}$, then

$$\rho\left(\sqrt{(\Phi(\mathbf{y}))^t\Phi(\mathbf{y})}\right) = \tilde{\rho}\left(\sqrt{(\Phi(\mathbf{y}))^tPDP^t\Phi(\mathbf{y})}\right). \tag{17}$$

Let v_i, $i = 1, 2, \ldots n$, denote the i-th column vector of P and d_i the i-th eigenvalue of A^tA. We set $\mathbf{y} = tv_i$, with $t > 0$, in (17). Thus, $\rho(t) = \tilde{\rho}(t\sqrt{d_1}) = \cdots = \tilde{\rho}(t\sqrt{d_n})$. Suppose for instance that $d_1 > d_2$ and set $\gamma = \sqrt{\frac{d_2}{d_1}}$. Then we have $\tilde{\rho}(t) = \tilde{\rho}(\gamma t) = \ldots = \tilde{\rho}(\gamma^m t)$ for all $t > 0$ and any $m = 1, 2, \ldots$. It follows that $\tilde{\rho}(t) = \tilde{\rho}(0^+)$. Therefore $\tilde{\rho}$ is a constant function, as are ρ and r. This contradicts **(A3)**. Thus, we have $d_1 = d_2$. In the same way we could show that $d_1 = d_2 = \cdots = d_n = a^2$. Consequently $\tilde{\rho}(t) = \rho(t/a)$ and $\tilde{\Phi}(\mathbf{x}) = aP\Phi(\mathbf{x})$.

□

As already pointed out, it is always possible to choose the initial value $\Phi(\mathbf{0})$ arbitrarily but $J_\Phi(\mathbf{0})$ cannot be chosen freely unlike in the *sr* case. Indeed, if $J_\Phi(\mathbf{0}) = J$ and $\tilde{\Phi}(\mathbf{x}) = aP\Phi(\mathbf{x})+b$ with $P \in S(n)$, then $J_{\tilde{\Phi}}(\mathbf{0}) = aPJ \in aS(n)J$, where $S(n)J$ is the equivalence class of J under right (in other words it is a right coset of $S(n)$ in $G(n)$). To choose one representative in $S(n)J$ we use the following lemma.

Lemma 3.2 *For any $J \in G(n)$, the right coset $S(n)J$ has a unique symmetric positive definite representative $(J^tJ)^{1/2}$.*

Proof. For any $B \in S(n)J$ there are some $P \in S(n)$ such that $B = PJ$. Thus $B^tB = J^tP^tPJ = J^tJ$ and it remains to show that $(J^tJ)^{1/2} \in S(n)J$. We have

$$(J^tJ)^{1/2}(J^tJ)^{-1}(J^tJ)^{1/2} = Id = \left((J^tJ)^{1/2}J^{-1}\right)\left((J^tJ)^{1/2}J^{-1}\right)^t,$$

then $(J^tJ)^{1/2}J^{-1} \in S(n)$.

□

Thus, without loss of generality we may impose the restriction

$$\begin{cases} \Phi(\mathbf{0}) = \mathbf{0} \ \text{ and } J_\Phi(\mathbf{0}) \text{ is a symmetric positive definite matrix} \\ \text{with } det(J_\Phi)(\mathbf{0}) = 1 \end{cases} \tag{18}$$

Conditions (18) mean nothing but we fix the homothetic Euclidean motions in $\mathbb{R}^n$.

Corollary 3.1 *Assume **(A1)**, **(A2)** and **(A3)** hold. If (Φ, ρ) is a solution to (1), with Φ satisfying conditions (18), it is unique.*

3.4 Form of the deformation.

We will solve equations (6) and (16) with respect to Φ. Set $J_\Phi^{-1}(\mathbf{0}) = (\gamma_{i,j})$ and $\alpha(\mathbf{x}) = (\alpha_1, \alpha_2, \ldots, \alpha_n)(\mathbf{x})$ with $\alpha_j(\mathbf{x}) = \sum_{i=1}^{n} \partial_i r(\mathbf{0}, \mathbf{x})\gamma_{i,j}$, $j = 1, 2, \ldots, n$. Here is our main theorem.

Theorem 3.3 *Assume **(A1)**, **(A2)** and **(A3)** are satisfied and suppose that (1) holds with a deformation Φ satisfying (18). Then*

$$\Phi(\mathbf{x}) = -\frac{\alpha(\mathbf{x})}{\|\alpha(\mathbf{x})\|}\sum_{i=1}^{n} x_i \int_0^1 \frac{\partial_i r(s\mathbf{x}, \mathbf{0})}{\|\alpha(s\mathbf{x})\|}ds.$$

Proof. Setting $\mathbf{x} = \mathbf{0}$ in (6) leads to

$$(\alpha(\mathbf{x}))^t J_\Phi(\mathbf{x}) = -D_\mathbf{y} r(\mathbf{0}, \mathbf{x}), \quad \mathbf{x} \neq \mathbf{0}, \tag{19}$$

and (16) is equivalent to

$$\Phi(\mathbf{x})(\alpha(\mathbf{x}))^t = \alpha(\mathbf{x})(\Phi(\mathbf{x}))^t. \tag{20}$$

We set $\partial_j \Phi(\mathbf{x}) = (\partial_j\phi_1, \partial_j\phi_2, \dots, \partial_j\phi_n)(\mathbf{x})$, $j = 1, 2, \dots, n$. It follows from (19) and (20) that

$$\alpha_i(\mathbf{x})((\Phi(\mathbf{x}))^t \partial_j \Phi(\mathbf{x})) = -\phi_i(\mathbf{x}) \partial_{n+j} r(\mathbf{0}, \mathbf{x}), \quad 1 \leq i, j \leq n.$$

For $\mathbf{x} \neq \mathbf{0}$ we have the relation $(\Phi(\mathbf{x}))^t \partial_j \Phi(\mathbf{x}) = \|\Phi(\mathbf{x})\| \partial_j(\|\Phi(\mathbf{x})\|)$, then

$$\alpha_i(\mathbf{x}) \|\Phi(\mathbf{x})\| \partial_j(\|\Phi(\mathbf{x})\|) = -\phi_i(\mathbf{x}) \partial_{n+j} r(\mathbf{0}, \mathbf{x}), \quad 1 \leq i, j \leq n.$$

It follows from (20) that $(\phi_i/\alpha_i)(\mathbf{x})$ is independent of i; then $(\phi_i/\alpha_i)(\mathbf{x}) = \|\Phi(\mathbf{x})\|/\|\alpha(\mathbf{x})\|$, $\Phi(\mathbf{x}) = \dfrac{\alpha(\mathbf{x})}{\|\alpha(\mathbf{x})\|}\|\Phi(\mathbf{x})\|$ and

$$\partial_i(\|\Phi(\mathbf{x})\|) = -\partial_{n+i} r(\mathbf{0}, \mathbf{x})/\|\alpha(\mathbf{x})\|, \quad i = 1, 2, \dots, n.$$

We set $\mathbf{x} = (x_1, x_2, \dots, x_n)$. By integration of the previous quantity we get

$$\|\Phi(\mathbf{x})\| = -\sum_{i=1}^n x_i \int_0^1 \frac{\partial_i r(s\mathbf{x}, \mathbf{0})}{\|\alpha(s\mathbf{x})\|} ds.$$

□

4 Discussion

So far, our work has been applied to correlation functions. In fact, provided that assumptions **(A2)** and **(A3)** are satisfied, it can be directly applied to every second order moment function like, for instance

- the dispersion functions $D(\mathbf{x}, \mathbf{y}) = Var(Z(\mathbf{x}) - Z(\mathbf{y}))$ that actually were the functions for which the model (1) had been introduced the first time (Sampson, 1986);
- the pair potential functions for Gibbs random fields (Jensen and Nielsen, 1998).

For relevance of the characterisation of the models (1) and (2), it is necessary that the functions under consideration have some parametric forms. The examples given above have such a parametric representation. But it is not the case in general for the second-order intensity functions and K-functions that are used in spatial point processes.

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